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Lower bounds for the eigenvalues of the basic Dirac operator on a Riemannian foliation

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Abstract

On a foliated Riemannian manifold with a transverse spin structure, we give a lower bound for the square of the eigenvalues of the basic Dirac operator by the smallest eigenvalue of the basic Yamabe operator. We prove, in the limiting case, that the foliation is minimal, transversally Einsteinian with constant transversal scalar curvature. In particular, if the codimension of $\mathcal{F}$ is $q = 3, 4, 7$ and 8, then $\mathcal{F}$ is transversally isometric to the action of discrete subgroup of $O(q)$ acting on the q-sphere.

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1. Introduction

The first estimate for the eigenvalues λ of the basic Dirac operator D_b restricted to the space of basic sections of a foliated spinor bundle on the foliated Riemannian manifold $(M, g_M, \mathcal{F})$ with a transverse spin structure was obtained by Jung [8]. Namely, by using a modified connection $\overset{f}{\nabla}$ defined by

$$\overset{f}{\nabla}_X \Psi = \nabla_X \Psi + f\pi(X) \cdot \Psi, \tag{1.1}$$

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one proved that the following inequality

$$\lambda^2 \geq \frac{q}{4(q-1)} \inf(\sigma^{\nabla} + |\kappa|^2) \tag{1.2}$$

holds, where $q = \mathrm{codim}\mathcal{F}$, σ^{∇} is the transversal scalar curvature and κ the mean curvature form of $\mathcal{F}$. In the limiting case, the foliation is minimal, transversally Einsteinian with constant transversal scalar curvature. The inequality (1.2) corresponds to those of Friedrich [4] and Hijazi [6].

In this paper, we give new lower bound for the eigenvalues of D_{b} by the smallest eigenvalue of the basic Yamabe operator Y_{b}, which is defined by

$$Y_{\mathrm{b}} = 4\frac{q-1}{q-2}\Delta_{\mathrm{B}} + \sigma^{\nabla}, \tag{1.3}$$

where Δ_{B} is a basic Laplacian acting on basic functions. The main idea used in this paper comes from Hijazi's paper [6]. This paper is organized as follows. In Section 2, we review the known facts on the foliated Riemannian manifold. In Section 3, we study the basic properties of the transversal Dirac operators of transversally conformally related metrics. In Section 4, we estimate the conformal lower bound for the eigenvalues of the basic Dirac operator. Let $(M, g_M, \mathcal{F})$ be a compact Riemannian manifold with a transverse spin foliation $\mathcal{F}$ of codimension $q \geq 3$ and bundle-like metric g_M such that $\kappa \in \Omega^1_{\mathrm{B}}(\mathcal{F})$ and $\delta\kappa = 0$. Then

$$\lambda^2 \geq \frac{q}{4(q-1)}(\mu_1 + \inf|\kappa|^2), \tag{1.4}$$

where μ_1 is the smallest eigenvalue of Y_{b}. This inequality is a specialization of the Hijazi inequality [6] to the case of Riemannian foliations. In Section 5, we prove, in the limiting case that $\mathcal{F}$ is minimal, transversally Einsteinian with positive constant transversal scalar curvature and there are no non-trivial parallel basic r-forms ($r \neq 0, q$) on M. Moreover, by the generalized Lichnerowicz and Obata theorem for foliations [12], we prove that in case of $q = 3, 4, 7$ and 8, $\mathcal{F}$ is transversally isometric to the space of orbits a discrete subgroup of $\mathrm{O}(q)$ acting on the standard q-sphere (see [7] for ordinary manifold).

Many steps and notations in this paper are similar to those of [6,7]. But we should take care of the equations containing the mean curvature form κ of $\mathcal{F}$.

Throughout this paper, we consider the bundle-like metric g_M for $(M, \mathcal{F})$ such that the mean curvature form κ is basic and harmonic. The existence of the bundle-like metric g_M for $(M, \mathcal{F})$ such that κ is basic, i.e., $\kappa \in \Omega^1_{\mathrm{B}}(\mathcal{F})$, is proved in [3]. In [13,14], for any bundle-like metric g_M with $\kappa \in \Omega^1_{\mathrm{B}}(\mathcal{F})$, it is proved that there exists another bundle-like metric $\tilde{g}_M$ for which the mean curvature form $\tilde{\kappa}$ is basic-harmonic.

2. Preliminaries and known facts

Let $(M, g_M, \mathcal{F})$ be a $(p+q)$-dimensional Riemannian manifold with a foliation $\mathcal{F}$ of codimension q and bundle-like metric g_M with respect to $\mathcal{F}$.

We recall the exact sequence

$$0 \to L \to TM \xrightarrow{\pi} Q \to 0$$

determined by the tangent bundle L and the normal bundle $Q = TM/L$ of $\mathcal{F}$. The assumption of g_M to be a bundle-like metric means that the induced metric g_Q on the normal bundle $Q \cong L^\perp$ satisfies the holonomy invariance condition $\overset{\circ}{\nabla} g_Q = 0$, where ∇° is the Bott connection in Q.

For a distinguished chart $\mathcal{U} \subset M$ the leaves of $\mathcal{F}$ in $\mathcal{U}$ are given as the fibers of a Riemannian submersion $f : \mathcal{U} \to \mathcal{V} \subset N$ onto an open subset $\mathcal{V}$ of a model Riemannian manifold N.

For overlapping charts $U_\alpha \cap U_\beta$, the corresponding local transition functions $\gamma_{\alpha\beta} = f_\alpha \circ f_\beta^{-1}$ on N are isometries. Further, we denote by ∇ the canonical connection of the normal bundle Q of $\mathcal{F}$. It is defined by

$$\nabla_X s = \pi([X, Y_s]) \quad \text{for } X \in \Gamma L, \qquad \nabla_X s = \pi(\nabla^M_X Y_s) \quad \text{for } X \in \Gamma L^\perp, \tag{2.1}$$

where $s \in \Gamma Q$ and $Y_s \in \Gamma L^\perp$ corresponding to s under the canonical isomorphism $Q \cong L^\perp$. The connection ∇ is metric and torsion free. It corresponds to the Riemannian connection of the model space N [9]. The curvature R^∇ of ∇ is defined by

$$R^\nabla(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} \quad \text{for } X, Y \in TM.$$

Since $i(X)R^\nabla = 0$ for any $X \in \Gamma L$ [9], we can define the (transversal) Ricci curvature $\rho^\nabla : \Gamma Q \to \Gamma Q$ and the (transversal) scalar curvature σ^∇ of $\mathcal{F}$ by

$$\rho^\nabla(s) = \sum_a R^\nabla(s, E_a)E_a, \qquad \sigma^\nabla = \sum_a g_Q(\rho^\nabla(E_a), E_a),$$

where $\{E_a\}_{a=1,\dots,q}$ is an orthonormal basis of Q. $\mathcal{F}$ is said to be (transversally) *Einsteinian* if the model space N is Einsteinian, that is,

$$\rho^\nabla = \frac{1}{q}\sigma^\nabla \cdot \mathrm{id} \tag{2.2}$$

with constant transversal scalar curvature σ^∇.

The *second fundamental form* of α of $\mathcal{F}$ is given by

$$\alpha(X, Y) = \pi(\nabla^M_X Y) \quad \text{for } X, Y \in \Gamma L. \tag{2.3}$$

It is trivial that α is Q-valued, bilinear and symmetric.

The *mean curvature vector field* of $\mathcal{F}$ is then defined by

$$\tau = \sum_i \alpha(E_i, E_i), \tag{2.4}$$

where $\{E_i\}_{i=1,\dots,p}$ is an orthonormal basis of L. The dual form κ, the *mean curvature form* for L, is then given by

$$\kappa(X) = g_Q(\tau, X) \quad \text{for } X \in \Gamma Q. \tag{2.5}$$

The foliation $\mathcal{F}$ is said to be *minimal* (or *harmonic*) if $\kappa = 0$. Throughout this paper, we also use the notation κ instead of the mean curvature vector τ.

Let $\Omega^r_B(\mathcal{F})$ be the space of all *basic r-forms*, i.e.,

$$\Omega^r_B(\mathcal{F}) = \{\phi \in \Omega^r(M) | i(X)\phi = 0, \theta(X)\phi = 0, \text{ for } X \in \Gamma L\}.$$

The foliation $\mathcal{F}$ is said to be *isoparametric* if $\kappa \in \Omega^1_B(\mathcal{F})$. We already know that κ is closed, i.e., $d\kappa = 0$ if $\mathcal{F}$ is isoparametric [16]. Since the exterior derivative preserves the basic forms (that is, $\theta(X)\, d\phi = 0$ and $i(X)\, d\phi = 0$ for $\phi \in \Omega^r_B(\mathcal{F})$), the restriction $d_B = d|_{\Omega^*_B(\mathcal{F})}$ is well defined. Let δ_B the adjoint operator of d_B. Then it is well-known [1,8] that

$$d_B = \sum_a \theta_a \wedge \nabla_{E_a}, \qquad \delta_B = -\sum_a \mathrm{i}(E_a)\nabla_{E_a} + \mathrm{i}(\kappa_B), \tag{2.6}$$

where κ_B is the basic component of κ, $\{E_a\}$ is a local orthonormal basic frame in Q and $\{\theta_a\}$ its g_Q-dual 1-form.

The *basic Laplacian* acting on $\Omega^*_B(\mathcal{F})$ is defined by

$$\Delta_B = d_B\delta_B + \delta_B d_B. \tag{2.7}$$

If $\mathcal{F}$ is the foliation by points of M, the basic Laplacian is the ordinary Laplacian. In the more general case, the basic Laplacian and its spectrum provide information about the transverse geometry of $(M, \mathcal{F})$ [15].

3. Transversal Dirac operators of transversally conformally related metrics

Let $(M, g_M, \mathcal{F})$ be a compact Riemannian manifold with a transverse spin foliation $\mathcal{F}$ of codimension q and a bundle-like metric g_M with respect to $\mathcal{F}$. Let $P_{so}(\mathcal{F})$ be the principal bundle of (oriented) transverse orthonormal framings. Then the transverse spin structure is a principal Spin(q)-bundle $P_{spin}(\mathcal{F})$ associated with it which is a fiberwise non-trivial double covering of $P_{so}(\mathcal{F})$. Let $S(\mathcal{F})$ be the foliated spinor bundle [5,8,10] associated with $P_{spin}(\mathcal{F})$. Then the transversal Dirac operator D_{tr} is locally defined [2,5] by

$$D_{tr}\Psi = \sum_a E_a \cdot \nabla_{E_a}\Psi - \frac{1}{2}\kappa \cdot \Psi \quad \text{for } \Psi \in \Gamma S(\mathcal{F}), \tag{3.1}$$

where $\{E_a\}$ is a local orthonormal basic frame of Q. We define the subspace $\Gamma_B(S(\mathcal{F}))$ of *basic* or *holonomy invariant* sections of $S(\mathcal{F})$ by

$$\Gamma_B(S(\mathcal{F})) = \{\Psi \in \Gamma S(\mathcal{F}) | \nabla_X \Psi = 0 \text{ for } X \in \Gamma L\}.$$

Trivially, we see that D_{tr} leaves $\Gamma_B(S(\mathcal{F}))$ invariant if and only if the foliation $\mathcal{F}$ is isoparametric, i.e., $\kappa \in \Omega^1_B(\mathcal{F})$. Let $D_b = D_{tr}|_{\Gamma_B(S(\mathcal{F}))} : \Gamma_B(S(\mathcal{F})) \to \Gamma_B(S(\mathcal{F}))$. This operator D_b is called the *basic Dirac operator* on (smooth) basic sections. On an isoparametric transverse spin foliation $\mathcal{F}$ with $\delta\kappa = 0$, it is well-known [2,5,8] that

$$D^2_{tr}\Psi = \nabla^*_{tr}\nabla_{tr}\Psi + \tfrac{1}{4}K^\nabla_\sigma \Psi, \tag{3.2}$$

where $K^\nabla_\sigma = \sigma^\nabla + |\kappa|^2$ and

$$\nabla^*_{tr}\nabla_{tr}\Psi = -\sum_a \nabla^2_{E_a,E_a}\Psi + \nabla_\kappa \Psi. \tag{3.3}$$

The operator $\nabla_{\mathrm{tr}}^{*}\nabla_{\mathrm{tr}}$ is non-negative and formally self-adjoint [8]. We now define a canonical section $\mathcal{R}^{\nabla}$ of $\mathrm{Hom}(S(\mathcal{F}), S(\mathcal{F}))$ by the formula

$$\mathcal{R}^{\nabla}(\Psi) = \sum_{a<b} E_a \cdot E_{\mathrm{b}} \cdot R^S(E_a, E_{\mathrm{b}})\Psi, \tag{3.4}$$

where the curvature transform R^S on $S(\mathcal{F})$ is given [11] as

$$R^S(X, Y)\Psi = \frac{1}{4}\sum_{a,b} g_Q(R^{\nabla}(X, Y)E_a, E_{\mathrm{b}})E_a \cdot E_{\mathrm{b}} \cdot \Psi \quad \text{for } X, Y \in \Gamma TM. \tag{3.5}$$

Lemma 3.1 (Jung [8]). *On the foliated spinor bundle $S(\mathcal{F})$, we have the following equations*

$$\mathcal{R}^{\nabla} = \tfrac{1}{4}\sigma^{\nabla}, \tag{3.6}$$

$$\sum_{a} E_a \cdot R^S(X, E_a)\Psi = -\frac{1}{2}\rho^{\nabla}(X) \cdot \Psi \quad \text{for } X \in \Gamma Q. \tag{3.7}$$

Now, we consider, for any real basic function u on M, the transversally conformal metric $\bar{g}_Q = \mathrm{e}^{2u} g_Q$. Let $\bar{P}_{\mathrm{so}}(\mathcal{F})$ be the principal bundle of $\bar{g}_Q$-orthogonal frames. Locally, the section $\bar{s}$ of $\bar{P}_{\mathrm{so}}(\mathcal{F})$ corresponding a section $s = (E_1, \ldots, E_q)$ of $P_{\mathrm{so}}(\mathcal{F})$ is $\bar{s} = (\bar{E}_1, \ldots, \bar{E}_q)$, where $\bar{E}_a = \mathrm{e}^{-u} E_a (a = 1, \ldots, q)$. This isometry will be denoted by I_u. Thanks to the isomorphism I_u one can define a transverse spin structure $\bar{P}_{\mathrm{spin}}(\mathcal{F})$ on $\mathcal{F}$ in such a way that the diagram commutes.

$$\begin{array}{ccc} P_{spin}(\mathcal{F}) & \xrightarrow{\tilde{I}_u} & \bar{P}_{spin}(\mathcal{F}) \\ \downarrow & & \downarrow \\ P_{so}(\mathcal{F}) & \xrightarrow{I_u} & \bar{P}_{so}(\mathcal{F}) \end{array}$$

Let $\bar{S}(\mathcal{F})$ be the foliated spinor bundles associated with $\bar{P}_{\mathrm{spin}}(\mathcal{F})$. For any section Ψ of $S(\mathcal{F})$, we write $\bar{\Psi} \equiv I_u\Psi$. If $\langle\,,\,\rangle_{g_Q}$ and $\langle\,,\,\rangle_{\bar{g}_Q}$ denote, respectively, the natural Hermitian metrics on $S(\mathcal{F})$ and $\bar{S}(\mathcal{F})$, then for any $\Phi, \Psi \in \Gamma S(\mathcal{F})$

$$\langle \Phi, \Psi\rangle_{g_Q} = \langle \bar{\Phi}, \bar{\Psi}\rangle_{\bar{g}_Q} \tag{3.8}$$

and the Clifford multiplication in $\bar{S}(\mathcal{F})$ is given by

$$\bar{X} \bar{\cdot} \bar{\Psi} = \overline{X \cdot \Psi} \quad \text{for } X \in \Gamma Q. \tag{3.9}$$

Let $\bar{\nabla}$ be the metric and torsion free connection corresponding to $\bar{g}_Q$. Then we have for $X, Y \in \Gamma TM$,

$$\bar{\nabla}_X \pi(Y) = \nabla_X \pi(Y) + X(u)\pi(Y) + Y(u)\pi(X) - g_Q(\pi(X), \pi(Y))\,\mathrm{grad}_{\nabla}(u), \tag{3.10}$$

where $\mathrm{grad}_{\nabla}(u) = \sum_a E_a(u)E_a$ is a transversal gradient of u and $X(u)$ is the Lie derivative of the function u in the direction of X. The formula (3.10) follows from that $\bar{\nabla}$ is the metric and torsion free connection with respect to $\bar{g}_Q$.

From (3.10), we have the following proposition.

Proposition 3.2. *The connection* ∇ *and* $\bar\nabla$ *acting respectively on the sections of* $S(\mathcal{F})$ *and* $\bar S(\mathcal{F})$, *are related, for any vector field* X *and any spinor field* Ψ *by*

$$\bar\nabla_X\bar\Psi = \overline{\nabla_X\Psi} - \tfrac{1}{2}\overline{\pi(X)\cdot \mathrm{grad}_\nabla(u)\cdot\Psi} - \tfrac{1}{2}g_Q(\mathrm{grad}_\nabla(u),\pi(X))\bar\Psi. \tag{3.11}$$

Proof. Let $\{E_a\}$ be an orthonormal basis of Q and denote by ω and $\bar\omega$, the connection forms corresponding to g_Q and $\bar g_Q$. That is, for any vector field $X\in TM$,

$$\nabla_X E_{\mathrm b} = \sum_c \omega_{bc}(\pi(X))E_c, \qquad \bar\nabla_X\bar E_{\mathrm b} = \sum_c \bar\omega_{bc}(\pi(X))\bar E_c. \tag{3.12}$$

From (3.10), we have

$$\bar\omega_{bc}(\pi(X)) = \omega_{bc}(\pi(X)) + g_Q(\pi(X),E_c)E_{\mathrm b}(u) - g_Q(\pi(X),E_{\mathrm b})E_c(u). \tag{3.13}$$

Let $\{\Psi_A\}(A=1,\dots,2^{[q/2]})$ be a local frame field of $S(\mathcal{F})$. Then the spinor covariant derivative of Ψ_A is given [11] by

$$\nabla_X\Psi_A = \frac{1}{2}\sum_{b<c}\omega_{bc}(\pi(X))E_{\mathrm b}\cdot E_c\cdot\Psi_A. \tag{3.14}$$

With respect to $\bar g_Q$, we have

$$\begin{aligned}
\bar\nabla_X\bar\Psi_A &= \frac{1}{2}\sum_{b<c}\bar\omega_{bc}(\pi(X))\bar E_{\mathrm b}\bar\cdot\bar E_c\bar\cdot\bar\Psi_A\\
&= \frac{1}{2}\sum_{b<c}\{\omega_{bc}(\pi(X)) + g_Q(\pi(X),E_c)E_{\mathrm b}(u) - g_Q(\pi(X),E_{\mathrm b})E_c(u)\}\bar E_{\mathrm b}\bar\cdot\bar E_c\bar\cdot\bar\Psi_A\\
&= \overline{\nabla_X\Psi_A} - \frac{1}{2}\sum_{b\neq c}g_Q(\pi(X),E_c)E_{\mathrm b}(u)\bar E_c\bar\cdot\bar E_{\mathrm b}\bar\cdot\bar\Psi_A\\
&= \overline{\nabla_X\Psi_A} - \frac{1}{2}\overline{\pi(X)\cdot\mathrm{grad}_\nabla(u)\cdot\Psi_A} - \frac{1}{2}g_Q(\mathrm{grad}_\nabla(u),\pi(X))\bar\Psi_A. \qquad\square
\end{aligned}$$

Let $\bar D_{\mathrm{tr}}$ be the transversal Dirac operator associated with the metric $\bar g_Q = \mathrm{e}^{2u}g_Q$ and acting on the sections of the foliated spinor bundle $\bar S(\mathcal{F})$. Let $\{E_a\}$ be a local frame of $P_{\mathrm{so}}(\mathcal{F})$ and $\{\bar E_a\}$ a local frame of $\bar P_{\mathrm{so}}(\mathcal{F})$. Locally, $\bar D_{\mathrm{tr}}$ is expressed by

$$\bar D_{\mathrm{tr}}\bar\Psi = \sum_a \bar E_a\bar\cdot\bar\nabla_{\bar E_a}\bar\Psi - \frac{1}{2}\kappa_{\bar g}\bar\cdot\bar\Psi, \tag{3.15}$$

where $\kappa_{\bar g}$ is the mean curvature form associated with $\bar g_Q$, which satisfies $\kappa_{\bar g} = \mathrm{e}^{-2u}\kappa$. Using (3.11), we have that for any Ψ,

$$\bar D_{\mathrm{tr}}\bar\Psi = \mathrm{e}^{-u}\{\overline{D_{\mathrm{tr}}\Psi} + \tfrac{1}{2}(q-1)\,\overline{\mathrm{grad}_\nabla(u)\cdot\Psi}\}. \tag{3.16}$$

Now, for any function f, we have $D_{\mathrm{tr}}(f\Psi) = \mathrm{grad}_\nabla(f)\cdot\Psi + fD_{\mathrm{tr}}\Psi$. Hence we have

$$\bar D_{\mathrm{tr}}(f\bar\Psi) = \mathrm{e}^{-u}\,\overline{\mathrm{grad}_\nabla(f)\cdot\Psi} + f\bar D_{\mathrm{tr}}\bar\Psi. \tag{3.17}$$

From (3.16) and (3.17), we have the following proposition.

Proposition 3.3. *Let $\mathcal{F}$ be the transverse spin foliation of codimension q. Then the transverse Dirac operators D_{tr} and $\bar{D}_{\text{tr}}$ satisfy*

$$\bar{D}_{\text{tr}}(e^{-((q-1)/2)u}\bar{\Psi}) = e^{-((q+1)/2)u}\overline{D_{\text{tr}}\Psi} \tag{3.18}$$

for any spinor field $\Psi \in S(\mathcal{F})$.

From Proposition 3.3, if $D_{\text{tr}}\Psi = 0$, then $\bar{D}_{\text{tr}}\bar{\Phi} = 0$, where $\Phi = e^{-((q-1)/2)u}\Psi$, and conversely. So we have the following corollary.

Corollary 3.4. *On the transverse spin foliation $\mathcal{F}$, the dimension of the space of the foliated harmonic spinors is a transversally conformal invariant.*

Let the mean curvature form κ of $\mathcal{F}$ be basic-harmonic, i.e., $\kappa \in \Omega_B^1(\mathcal{F})$ and $\delta_B\kappa = 0$. Then by direct calculation, we have the Lichnerowicz type formula

$$\bar{D}_{\text{tr}}^2\bar{\Psi} = \bar{\nabla}_{\text{tr}}^*\bar{\nabla}_{\text{tr}}\bar{\Psi} + \mathcal{R}^{\bar{\nabla}}(\bar{\Psi}) + K^{\bar{\nabla}}\bar{\Psi}, \tag{3.19}$$

where

$$\bar{\nabla}_{\text{tr}}^*\bar{\nabla}_{\text{tr}}\bar{\Psi} = -\sum_a \bar{\nabla}_{\bar{E}_a}\bar{\nabla}_{\bar{E}_a}\bar{\Psi} + \bar{\nabla}_{\sum \bar{\nabla}_{\bar{E}_a}\bar{E}_a}\bar{\Psi} + \bar{\nabla}_{\kappa_{\bar{g}}}\bar{\Psi}, \tag{3.20}$$

$$K^{\bar{\nabla}} = \tfrac{1}{2}(q-2)\kappa_{\bar{g}}(u) + \tfrac{1}{4}|\bar{\kappa}|^2, \tag{3.21}$$

$$\mathcal{R}^{\bar{\nabla}}(\bar{\Psi}) = \sum_{a<b} \bar{E}_a \cdot \bar{E}_b \cdot \bar{R}^S(\bar{E}_a, \bar{E}_b)\bar{\Psi}. \tag{3.22}$$

Lemma 3.5. *Let $(M, g_M, \mathcal{F})$ be a compact Riemannian manifold with a foliation $\mathcal{F}$ and a bundle-like metric g_M with respect to $\mathcal{F}$. Then*

$$\langle\langle \bar{\nabla}_{\text{tr}}^*\bar{\nabla}_{\text{tr}}\bar{\Psi}, \bar{\Phi}\rangle\rangle_{\bar{g}_Q} = \langle\langle \bar{\nabla}_{\text{tr}}\bar{\Psi}, \bar{\nabla}_{\text{tr}}\bar{\Phi}\rangle\rangle_{\bar{g}_Q}$$

for all $\Phi, \Psi \in S(\mathcal{F})$, where $\langle \bar{\nabla}_{\text{tr}}\bar{\Psi}, \bar{\nabla}_{\text{tr}}\bar{\Phi}\rangle_{\bar{g}_Q} = \sum_a \langle \bar{\nabla}_{\bar{E}_a}\bar{\Psi}, \bar{\nabla}_{\bar{E}_a}\bar{\Phi}\rangle_{\bar{g}_Q}$.

Proof. Fix $x \in M$ and choose an orthonormal basic frame $\{E_a\}$ such that $(\nabla E_a)_x = 0$ for all a. Then we have that at x

$$\bar{\nabla}_{\bar{E}_a}\bar{E}_b = e^{-2u}\{E_b(u)E_a - \delta_{ab}\,\text{grad}_\nabla(u)\}. \tag{3.23}$$

Hence we have

$$\begin{aligned}
\langle \bar{\nabla}_{\text{tr}}^*\bar{\nabla}_{\text{tr}}\bar{\Psi}, \bar{\Phi}\rangle_{\bar{g}_Q} &= -\sum_a \langle \bar{\nabla}_{\bar{E}_a}\bar{\nabla}_{\bar{E}_a}\bar{\Psi}, \bar{\Phi}\rangle_{\bar{g}_Q} + (1-q)\,e^{-2u} \\
&\quad \times \langle \bar{\nabla}_{\text{grad}_\nabla(u)}\bar{\Psi}, \bar{\Phi}\rangle_{\bar{g}_Q} + \langle \bar{\nabla}_{\kappa_{\bar{g}}}\bar{\Psi}, \bar{\Phi}\rangle_{\bar{g}_Q} \\
&= -\sum_a \bar{E}_a\langle \bar{\nabla}_{\bar{E}_a}\bar{\Psi}, \bar{\Phi}\rangle_{\bar{g}_Q} + \sum_a \langle \bar{\nabla}_{\bar{E}_a}\bar{\Psi}, \bar{\nabla}_{\bar{E}_a}\bar{\Phi}\rangle_{\bar{g}_Q} + (1-q)\,e^{-2u} \\
&\quad \times \langle \bar{\nabla}_{\text{grad}_\nabla(u)}\bar{\Psi}, \bar{\Phi}\rangle_{\bar{g}_Q} + \langle \bar{\nabla}_{\kappa_{\bar{g}}}\bar{\Psi}, \bar{\Phi}\rangle_{\bar{g}_Q} \\
&= -\text{div}_{\bar{\nabla}}(V) + \sum_a \langle \bar{\nabla}_{\bar{E}_a}\bar{\Psi}, \bar{\nabla}_{\bar{E}_a}\bar{\Phi}\rangle_{\bar{g}_Q} + \langle \bar{\nabla}_{\kappa_{\bar{g}}}\bar{\Psi}, \bar{\Phi}\rangle_{\bar{g}_Q},
\end{aligned}$$

where $V \in \Gamma Q \otimes \mathbb{C}$ are defined by $\bar{g}_Q(V, Z) = \langle \bar{\nabla}_Z \bar{\Psi}, \bar{\Phi} \rangle_{\bar{g}_Q}$ for all $Z \in \Gamma Q$. The last line is proved as follows: at $x \in M$,

$$\begin{aligned}\mathrm{div}_{\bar{\nabla}}(V) &= \sum_a \bar{g}_Q(\bar{\nabla}_{\bar{E}_a} V, \bar{E}_a) = \sum_a \bar{E}_a \bar{g}_Q(V, \bar{E}_a) - \bar{g}_Q\left(V, \sum_a \bar{\nabla}_{\bar{E}_a} \bar{E}_a\right)\\ &= \sum_a \bar{E}_a \langle \bar{\nabla}_{\bar{E}_a} \bar{\Psi}, \bar{\Phi} \rangle_{\bar{g}_Q} - (1-q)\,\mathrm{e}^{-2u} \langle \bar{\nabla}_{\mathrm{grad}_\nabla(u)} \bar{\Psi}, \bar{\Phi} \rangle_{\bar{g}_Q}.\end{aligned}$$

By Green's theorem on the foliated Riemannian manifold [17]

$$\int_M \mathrm{div}_{\bar{\nabla}}(V) v_{\bar{g}} = \int_M \bar{g}_Q(\kappa_{\bar{g}}, V) v_{\bar{g}} = \int_M \langle \bar{\nabla}_{\kappa_{\bar{g}}} \bar{\Psi}, \bar{\Phi} \rangle_{\bar{g}_Q} v_{\bar{g}},$$

where $v_{\bar{g}}$ is the volume form associated to the metric $\bar{g}_M = g_L + \bar{g}_Q$. By integrating, we obtain our result. □

4. Eigenvalue estimate of the basic Dirac operator

Let $(M, g_M, \mathcal{F})$ be a Riemannian manifold with a transversally oriented Riemannian spin foliation $\mathcal{F}$ of codimension $q \geq 2$. Let g_M be the bundle-like metric for which the mean curvature κ is basic-harmonic, i.e., $\kappa \in \Omega^1_{\mathrm{B}}(\mathcal{F})$ and $\delta_{\mathrm{B}} \kappa = 0$.

Now, we introduce a new connection $\overset{f}{\bar{\nabla}}$ on $\bar{S}(\mathcal{F})$ as

$$\overset{f}{\bar{\nabla}}_X \bar{\Psi} = \bar{\nabla}_X \bar{\Psi} + f\pi(X)\bar{\cdot}\bar{\Psi} \quad \text{for } X \in TM, \tag{4.1}$$

where f is a real-valued basic function on M and $\pi : TM \to Q$. Trivially, this connection $\overset{f}{\bar{\nabla}}$ is a metric connection.

Lemma 4.1. *On the foliated spinor bundle $\bar{S}(\mathcal{F})$, we have*

$$\langle\langle \overset{f}{\bar{\nabla}}{}^*_{\mathrm{tr}} \overset{f}{\bar{\nabla}}_{\mathrm{tr}} \bar{\Psi}, \bar{\Phi} \rangle\rangle_{\bar{g}_Q} = \langle\langle \overset{f}{\bar{\nabla}}_{\mathrm{tr}} \bar{\Psi}, \overset{f}{\bar{\nabla}}_{\mathrm{tr}} \bar{\Phi} \rangle\rangle_{\bar{g}_Q}$$

for all $\Psi, \Phi \in \Gamma S(\mathcal{F})$, where $\langle \overset{f}{\bar{\nabla}}_{\mathrm{tr}} \bar{\Psi}, \overset{f}{\bar{\nabla}}_{\mathrm{tr}} \bar{\Phi} \rangle_{\bar{g}_Q} = \sum_a \langle \overset{f}{\bar{\nabla}}_{\bar{E}_a} \bar{\Psi}, \overset{f}{\bar{\nabla}}_{\bar{E}_a} \bar{\Phi} \rangle_{\bar{g}_Q}$.

Proof. The proof is similar to the one in Lemma 3.5. □

On the other hand, by using (3.20), (3.23) and (4.1), we have

$$\overset{f}{\bar{\nabla}}{}^*_{\mathrm{tr}} \overset{f}{\bar{\nabla}}_{\mathrm{tr}} \bar{\Psi} = \bar{\nabla}^*_{\mathrm{tr}} \bar{\nabla}_{\mathrm{tr}} \bar{\Psi} - 2f \bar{D}_{\mathrm{tr}} \bar{\Psi} + qf^2 \bar{\Psi} - \mathrm{e}^{-u}\, \overline{\mathrm{grad}_\nabla(f) \cdot \Psi}. \tag{4.2}$$

From (3.19), we have

$$\overset{f}{\bar{\nabla}}{}^*_{\mathrm{tr}} \overset{f}{\bar{\nabla}}_{\mathrm{tr}} \bar{\Psi} = \bar{D}^2_{\mathrm{tr}} \bar{\Psi} - 2f \bar{D}_{\mathrm{tr}} \bar{\Psi} - \mathcal{R}^{\bar{\nabla}}(\bar{\Psi}) + qf^2 \bar{\Psi} - K^{\bar{\nabla}} \bar{\Psi} - \mathrm{e}^{-u}\, \overline{\mathrm{grad}_\nabla(f) \cdot \Psi}, \tag{4.3}$$

where $K^{\bar\nabla} = (1/2)\{(q-2)\kappa_{\bar g}(u) + (1/2)|\bar\kappa|^2\}$. By integrating (4.3), we have

$$\int |\overset{f}{\bar\nabla}_{\rm tr}\bar\Psi|^2_{\bar g_Q} = \int \langle \bar D^2_{\rm tr}\bar\Psi, \bar\Psi\rangle_{\bar g_Q} - 2\int f\langle \bar D_{\rm tr}\bar\Psi, \bar\Psi\rangle_{\bar g_Q} - \int \langle \mathcal{R}^{\bar\nabla}(\bar\Psi), \bar\Psi\rangle_{\bar g_Q} - \int (K^{\bar\nabla} - qf^2)|\bar\Psi|^2_{\bar g_Q} - \int {\rm e}^{-u}\langle \overline{{\rm grad}_\nabla(f)\cdot\Psi}, \bar\Psi\rangle_{\bar g_Q}. \tag{4.4}$$

Let $D_{\rm b}\Phi = \lambda\Phi(\Phi \neq 0)$. From Proposition 3.3, we have

$$\bar D_{\rm b}\bar\Psi = \lambda\,{\rm e}^{-u}\bar\Psi, \tag{4.5}$$

where $\bar\Psi = {\rm e}^{-((q-1)/2)u}\Phi$. From Lemma 3.1, we have $\mathcal{R}^{\bar\nabla} = (1/4)\sigma^{\bar\nabla}$, where $\sigma^{\bar\nabla}$ is the transversal scalar curvature of the metric $\bar g_Q = {\rm e}^{2u}g_Q$. Note that for all $X \in \Gamma Q$ and $\Psi \in \Gamma S(\mathcal{F})$, $\langle \overline{X\cdot\Psi}, \bar\Psi\rangle_{\bar g_Q}$ is purely imaginary [8]. Hence $\langle \overline{{\rm grad}_\nabla(f)\cdot\Psi}, \bar\Psi\rangle_{\bar g_Q}$ is purely imaginary. Hence we have

$$\int |\overset{f}{\bar\nabla}_{\rm tr}\bar\Psi|^2_{\bar g_Q} = \int \left(\lambda^2\,{\rm e}^{-2u} - 2f\lambda\,{\rm e}^{-u} - \frac{1}{4}\sigma^{\bar\nabla} - K^{\bar\nabla} + qf^2\right)|\bar\Psi|^2_{\bar g_Q}. \tag{4.6}$$

If we put $f = (\lambda/q)\,{\rm e}^{-u}$, then we have

$$\int |\overset{f}{\bar\nabla}_{\rm tr}\bar\Psi|^2_{\bar g_Q} = \frac{q-1}{q}\int {\rm e}^{-2u}\left(\lambda^2 - \frac{q}{4(q-1)}{\rm e}^{2u}K^{\bar\nabla}_\sigma\right)|\bar\Psi|^2_{\bar g_Q}, \tag{4.7}$$

where $K^{\bar\nabla}_\sigma = \sigma^{\bar\nabla} + 4K^{\bar\nabla}$. Hence we have the following theorem.

Theorem 4.2. *Let $(M, g_M, \mathcal{F})$ be a compact Riemannian manifold with a transverse spin foliation $\mathcal{F}$ of codimension $q \geq 2$ and bundle-like metric g_M such that $\kappa \in \Omega^1_{\rm B}(\mathcal{F})$ and $\delta\kappa = 0$. Assume that $K^{\bar\nabla}_\sigma \geq 0$ for some transversally conformal metric $\bar g_Q = {\rm e}^{2u}g_Q$. Then we have*

$$\lambda^2 \geq \frac{q}{4(q-1)}\sup_u \inf_M ({\rm e}^{2u}K^{\bar\nabla}_\sigma). \tag{4.8}$$

The transversal Ricci curvature $\rho^{\bar\nabla}$ of $\bar g_Q = {\rm e}^{2u}g_Q$ and the transversal scalar curvature $\sigma^{\bar\nabla}$ of $\bar g_Q$ are related to the transversal Ricci curvature ρ^∇ of g_Q and the transversal scalar curvature σ^∇ of g_Q by the following lemma.

Lemma 4.3. *On a Riemannian foliation $\mathcal{F}$, we have that for any $X \in Q$,*

$$\begin{aligned}{\rm e}^{2u}\rho^{\bar\nabla}(X) = {} & \rho^\nabla(X) + (2-q)\nabla_X\,{\rm grad}_\nabla(u) + (2-q)|{\rm grad}_\nabla(u)|^2X \\ & + (q-2)X(u)\,{\rm grad}_\nabla(u) + \{\Delta_{\rm B}u - \kappa(u)\}X,\end{aligned} \tag{4.9}$$

$${\rm e}^{2u}\sigma^{\bar\nabla} = \sigma^\nabla + (q-1)(2-q)|{\rm grad}_\nabla(u)|^2 + 2(q-1)\{\Delta_{\rm B}u - \kappa(u)\}. \tag{4.10}$$

From (4.10), we have

$${\rm e}^{2u}K^{\bar\nabla}_\sigma = \sigma^\nabla + |\kappa|^2 + 2(q-1)\Delta_{\rm B}u + (q-1)(2-q)|{\rm grad}_\nabla(u)|^2 - 2\kappa(u). \tag{4.11}$$

On the other hand, for $q \geq 3$, if we choose the positive function h by $u = 2/(q-2)\ln h$, then we have

$$\Delta_B u = \frac{2}{q-2}\{h^{-2}|\mathrm{grad}_\nabla(h)|^2 + h^{-1}\Delta_B h\}, \tag{4.12}$$

$$|\mathrm{grad}_\nabla(u)|^2 = \left(\frac{2}{q-2}\right)^2 h^{-2}|\mathrm{grad}_\nabla(h)|^2. \tag{4.13}$$

Hence we have

$$\mathrm{e}^{2u}K_\sigma^{\bar{\nabla}} = h^{(4/(q-2))}K_\sigma^{\bar{\nabla}} = h^{-1}Y_b h + |\kappa|^2 - \frac{4}{q-2}h^{-1}\kappa(h), \tag{4.14}$$

where

$$Y_b = 4\frac{q-1}{q-2}\Delta_B + \sigma^\nabla, \tag{4.15}$$

which is called a *basic Yamabe operator* of $\mathcal{F}$.

Now we put $\mathcal{K}_u = \{u \in \Omega_B^0(\mathcal{F}) | \kappa(u) = 0\}$. If we choose $u \in \mathcal{K}_u$, then $\kappa(h) = 0 = \kappa(u)$. From (4.11) and (4.14), we have

$$\mathrm{e}^{2u}K_\sigma^{\bar{\nabla}} = K_\sigma^\nabla + 2(q-1)\Delta_B u = h^{-1}Y_b h + |\kappa|^2, \tag{4.16}$$

where $K_\sigma^\nabla = \sigma^\nabla + |\kappa|^2$. Hence we have the following corollary.

Corollary 4.4. *Let $(M, g_M, \mathcal{F})$ be a compact Riemannian manifold with a transverse spin foliation $\mathcal{F}$ and bundle-like metric g_M such that $\kappa \in \Omega_B^1(\mathcal{F})$ and $\delta\kappa = 0$. Assume that $K_\sigma^\nabla \geq 0$. Then*

$$\lambda^2 \geq \begin{cases} \dfrac{q}{4(q-1)}\sup_{u\in\mathcal{K}_u}\inf_M\{K_\sigma^\nabla + 2(q-1)\Delta_B u + (q-1)(q-2)|\mathrm{grad}_\nabla(u)|^2\} & \text{if } q \geq 2, \\ \dfrac{q}{4(q-1)}\sup_{h\in\mathcal{K}_u}\inf_M\{h^{-1}Y_b h + |\kappa|^2\} & \text{if } q \geq 3. \end{cases} \tag{4.17}$$

Assume that the transversal scalar curvature σ^∇ is non-negative. Then the eigenvalue h_1 associated to the first eigenvalue μ_1 of Y_b can be chosen to be positive and then μ_1 is non-negative. Thus

$$h_1^{-1}Y_b h_1 = \mu_1. \tag{4.18}$$

Since $\sup\inf\{h^{-1}Y_b h\} \geq \mu_1$, we have the following corollary.

Corollary 4.5. *Let $(M, g_M, \mathcal{F})$ be a compact Riemannian manifold with a transverse spin foliation $\mathcal{F}$ of codimension $q \geq 3$ and bundle-like metric g_M with $\kappa \in \Omega_B^1(\mathcal{F})$ and $\delta\kappa = 0$. If the transversal scalar curvature satisfies $\sigma^\nabla \geq 0$, then we have*

$$\lambda^2 \geq \frac{q}{4(q-1)}(\mu_1 + \inf|\kappa|^2). \tag{4.19}$$

Remark. Since $\mu_1 \geq \inf\sigma^{\nabla}$, the inequality (4.19) is a sharper estimate than the previous one (1.2). Moreover, Corollary 4.5 is a specialization of the result on an ordinary manifold by Hijazi [6] to the case of Riemannian foliations.

5. The limiting case

In this section, we study the foliated Riemannian manifold M which admits a non-zero foliated spinor Ψ_1 such that $D_b\Psi_1 = \lambda_1\Psi_1$ with $\lambda_1^2 = (1/4)(q/(q-1))(\mu_1 + \inf|\kappa|^2)$. We define $\mathrm{Ric}^f_\nabla : \Gamma Q \otimes S(\mathcal{F}) \to S(\mathcal{F})$ by

$$\mathrm{Ric}^f_\nabla(X \otimes \Psi) = \sum_a E_a \cdot R^f(X, E_a)\Psi, \tag{5.1}$$

where R^f is the curvature tensor with respect to $\overset{f}{\nabla}$ defined by $\nabla^f_X\Psi = \nabla_X\Psi + f\pi(X)\cdot\Psi$. By long calculation, for $X \in \Gamma Q$ and $\Psi \in \Gamma S(\mathcal{F})$ we have [8]

$$\mathrm{Ric}^f_\nabla(X \otimes \Psi) = -\tfrac{1}{2}\rho^{\nabla}(X)\cdot\Psi + 2(q-1)f^2X\cdot\Psi - qX(f)\Psi - \mathrm{grad}_\nabla(f)\cdot X\cdot\Psi \tag{5.2}$$

for $X \in \Gamma Q$. Similarly, we obtain the formula for $\mathrm{Ric}^f_{\bar\nabla}(X \otimes \bar\Psi)$ associated to $\bar S(\mathcal{F})$. Namely,

$$\mathrm{Ric}^f_{\bar\nabla}(X \otimes \bar\Psi) = -\tfrac{1}{2}\rho^{\bar\nabla}(X)\bar\cdot\bar\Psi + 2(q-1)f^2X\bar\cdot\bar\Psi - qX(f)\bar\Psi - \overline{\mathrm{grad}_\nabla(f)}\bar\cdot X\bar\cdot\bar\Psi, \tag{5.3}$$

where $\rho^{\bar\nabla}(X)$ is the transversal Ricci curvature with respect to $\bar\nabla$. From (5.2) and (5.3), we have the following facts.

Proposition 5.1. *If M admits a non-zero foliated spinor Ψ with $\overset{f}{\bar\nabla}\bar\Psi = 0$, then f is constant and for any $X \in TM$*

$$\nabla_X\Psi = -f\,\mathrm{e}^u\pi(X)\cdot\Psi + \tfrac{1}{2}\pi(X)\cdot\mathrm{grad}_\nabla(u)\cdot\Psi + \tfrac{1}{2}g_Q(\mathrm{grad}_\nabla(u), \pi(X))\Psi. \tag{5.4}$$

Proof. From (5.3), it follows that f is constant (see [8]). Next, we have from (4.1)

$$\bar\nabla_X\bar\Psi + f\pi(X)\cdot\bar\Psi = 0 \quad \text{for any } X \in TM.$$

Hence from (3.11), we have

$$\overline{\nabla_X\Psi} - \tfrac{1}{2}\overline{\pi(X)\cdot\mathrm{grad}_\nabla(u)\cdot\Psi} - \tfrac{1}{2}g_Q(\mathrm{grad}_\nabla(u), \pi(X))\bar\Psi + f\,\mathrm{e}^u\overline{\pi(X)\cdot\Psi} = 0.$$

Since $\tilde I_u$ is an isometry, we have

$$\nabla_X\Psi - \tfrac{1}{2}\pi(X)\cdot\mathrm{grad}_\nabla(u)\cdot\Psi - \tfrac{1}{2}g_Q(\mathrm{grad}_\nabla(u), \pi(X))\Psi + f\,\mathrm{e}^u\pi(X)\cdot\Psi = 0.$$

This complete our proof. □

We now consider the limiting case. Let $\lambda_1^2 = (1/4)(q/(q-1))(\mu_1 + \inf|\kappa|^2)$. From (4.7), we have

$$\overset{f}{\bar{\nabla}}_{\mathrm{tr}}\bar{\Psi} = 0 \quad \text{with} \quad f = \frac{\lambda_1}{q}\mathrm{e}^{-u}, \qquad \mu_1 + \inf|\kappa|^2 = h^{-1}Y_{\mathrm{b}}h + |\kappa|^2. \tag{5.5}$$

By Proposition 5.1, we know that $f = (\lambda_1/q)\,\mathrm{e}^{-u}$ is constant. So u is constant. From (5.4), Ψ is a transversal Killing spinor, i.e., $\overset{f}{\bar{\nabla}}_{\mathrm{tr}}\Psi = 0$. Also, we have from (5.3)

$$\rho^{\bar{\nabla}}(X) = 4(q-1)f^2X \quad \text{for } X \in \Gamma Q. \tag{5.6}$$

Since u is constant, we have from (4.9)

$$\rho^{\nabla}(X) = \frac{4(q-1)}{q^2}\lambda_1^2 X. \tag{5.7}$$

If we compare (5.7) with (2.2), then $\mathcal{F}$ is transversally Einsteinian with a constant transversal scalar curvature $\sigma^{\nabla} = (4(q-1)/q)\lambda_1^2$. Since u is constant, h is constant. Hence the second equation in (5.5) with $\sigma^{\nabla} = (4(q-1)/q)\lambda_1^2$ implies that

$$\sigma^{\nabla} = \sigma^{\nabla} + |\kappa|^2.$$

Hence $|\kappa| = 0$. So $\mathcal{F}$ is minimal. Summing up, we have the following theorem.

Theorem 5.2. *Let $(M, g_M, \mathcal{F})$ be a compact Riemannian manifold with a transverse spin foliation $\mathcal{F}$ of codimension $q \geq 3$ and bundle-like metric g_M such that $\kappa \in \Omega_{\mathrm{B}}^1(\mathcal{F})$ and $\delta\kappa = 0$. Assume that $\sigma^{\nabla} \geq 0$. If there exists an eigenspinor field Ψ_1 of the basic Dirac operator D_{b} for the eigenvalue $\lambda_1^2 = (q/4(q-1))(\mu_1 + \inf|\kappa|^2)$, then Ψ_1 is a transversal Killing spinor and $\mathcal{F}$ is minimal, transversally Einsteinian with positive constant transversal scalar curvature σ^{∇}.*

Let $\omega \in \Omega_{\mathrm{B}}^r(\mathcal{F})$ be the basic r-form and $\Psi \in \Gamma_{\mathrm{B}}(S(\mathcal{F}))$ a foliated spinor field. Then we have from (2.6),

$$D_{\mathrm{b}}(\omega\cdot\Psi) = (d_{\mathrm{B}}\omega + \delta_{\mathrm{B}}\omega)\cdot\Psi + \sum_a E_a\cdot\omega\cdot\nabla_{E_a}\Psi - \frac{1}{2}\kappa\cdot\omega\cdot\Psi - \mathrm{i}(\kappa)\omega\cdot\Psi, \tag{5.8}$$

where $\{E_a\}$ is an orthonormal basis of Q. From Proposition 5.1, we have

$$\nabla_X\Psi_1 = -\frac{\lambda_1}{q}\pi(X)\cdot\Psi_1. \tag{5.9}$$

Moreover, for any basic r-form $\omega \in \Omega_{\mathrm{B}}^r(\mathcal{F})$, we have, by direct calculation,

$$\sum_a E_a\cdot\omega\cdot E_a = (-1)^{r-1}(q-2r)\omega. \tag{5.10}$$

From (5.4), (5.9) and (5.10), we have

$$D_{\mathrm{b}}(\omega\cdot\Psi)=(d_{\mathrm{B}}\omega+\delta_{\mathrm{B}}\omega)\cdot\Psi+(-1)^r\frac{(q-2r)\lambda_1}{q}\omega\cdot\Psi-\frac{1}{2}\kappa\cdot\omega\cdot\Psi-\mathrm{i}(\kappa)\omega\cdot\Psi. \tag{5.11}$$

Hence we have the following theorem.

Theorem 5.3 (cf. Hijazi [6]). *Let $(M, g_M, \mathcal{F})$ be a Riemannian manifold with a transverse spin foliation $\mathcal{F}$ of codimension $q \geq 3$ and a bundle-like metric g_M such that $\kappa \in \Omega^1_{\mathrm{B}}(\mathcal{F})$ and $\delta\kappa = 0$. Assume that $\sigma^\nabla \geq 0$. If there exists Ψ such that $D_{\mathrm{b}}\Psi = \lambda_1\Psi$ with $\lambda_1^2 = (q/4(q-1))(\mu_1 + \inf|\kappa|^2)$, then there are no non-trivial parallel basic r-forms $(r \neq 0, q)$ on M.*

Proof. By Theorem 5.2, $\mathcal{F}$ is minimal. Hence we have

$$D_{\mathrm{b}}(\omega\cdot\Psi)=(d_{\mathrm{B}}\omega+\delta_{\mathrm{B}}\omega)\Psi+(-1)^r\frac{q-2r}{q}\lambda_1\omega\cdot\Psi.$$

Assume that any basic r-form $\omega \in \Omega^r_{\mathrm{B}}(\mathcal{F})$ is parallel. Then $d_{\mathrm{B}}\omega = \delta_{\mathrm{B}}\omega = 0$. Hence we have

$$D_{\mathrm{b}}(\omega\cdot\Psi)=(-1)^r\frac{q-2r}{q}\lambda_1\omega\cdot\Psi.$$

So $\omega\cdot\Psi$ is a eigenspinor with eigenvalue $(-1)^r((q-2r)/q)\lambda_1$. If $r \neq 0$ and q, then its absolute value less that $|\lambda_1|$. This is contradiction. So we have

$$\omega\cdot\Psi = 0. \tag{5.12}$$

It follows by differentiation

$$\omega\cdot e_a\cdot\Psi = 0. \tag{5.13}$$

On the other hand, we know that for any 1-form θ and r-form ω,

$$\omega\cdot\theta = (-1)^r\{\theta\cdot\omega + 2\mathrm{i}(\theta^\sharp)\omega\}, \tag{5.14}$$

where $\theta^\sharp$ is a g_Q-dual vector of θ. From (5.12)–(5.14), we have

$$\mathrm{i}(e_a)\omega\cdot\Psi = 0.$$

After new differentiation, we get

$$\omega(e_{a_1},\ldots,e_{a_r})\Psi = 0 \tag{5.15}$$

which implies $\omega = 0$. This completes our proof. □

If $\mathcal{F}$ is minimal, then any parallel basic forms are harmonic. Hence we have the following corollary.

Corollary 5.4. *Under the same condition as in Theorem 5.3, if there exists Ψ such that $D_{\mathrm{b}}\Psi = \lambda_1\Psi$ with $\lambda_1^2 = (q/4(q-1))(\mu_1 + \inf|\kappa|^2)$, then there are no non-trivial basic harmonic r-forms* $(r \neq 0, q)$ *on* M.

Now, we recall the generalized Lichnerowicz–Obata theorem by Lee and Richardson for foliations [12].

Definition 5.5. Let G be a discrete group. Then $\mathcal{F}$ is *transversally isometric* to the isometric action of G on a Riemannian manifold N if there exists a smooth, surjective map $\phi : M \to N$ such that:

1. The function ϕ induces a homeomorphism between the leaf spact $M/\mathcal{F}$ and the orbit space N/G.
2. For each $x \in M$, the push forward ϕ_* restricts to an isometry $\phi_* : Q_x \to T_{\phi(x)}N$, where Q is the normal bundle of the foliation and TN is the tangent bundle of N.

Theorem 5.6 (Lee and Richardson's [12] generalized Lichnerowicz theorem). *Let $\mathcal{F}$ be a codimension q Riemannian foliation on a closed, connected Riemannian manifold M. Suppose that there exists a positive constant c such that the transversal Ricci curvature satisfies $\rho^{\nabla}(X) \geq c(q-1)X$ for every $X \in Q$. Then the smallest non-zero eigenvalue λ_{B} of the basic Laplacian Δ_{B} satisfies*

$$\lambda_{\mathrm{B}} \geq cq.$$

Theorem 5.7 (Lee and Richardson's [12] generalized Obata theorem). *The equality holds in Theorem 5.6 if and only if*:

1. *$\mathcal{F}$ is transversally isometric to the action of a discrete subgroup of* $\mathrm{O}(q)$ *acting on the q sphere of constant curvature c. Thus, there are at least two closed leaves* (*the poles*).
2. *If we choose the metric on M so that the mean curvature form is basic, then the mean curvature of the foliation is zero.*
3. *Each level set of the λ_{B} eigenfunction is the set of leaves corresponding to a latitude of the q sphere, and the volume $V(r)$ of this level set is the volume of the maximum leaf L times the volume of the latitude.*

For the classification of real Clifford algebra $\mathrm{Cl}(n)$ of $\mathbb{R}^n$, we have the following proposition.

Proposition 5.8 (Lawson and Michelsohn [11]). *For* $1 \leq n \leq 8$, *the Clifford algebra* $\mathrm{Cl}(n)$ *and the dimension d_n of an irreducible $\mathbb{R}$-module for* $\mathrm{Cl}(n)$ *are given by the following*:

$$\mathrm{Cl}(1) \cong \mathbb{C}, \quad \mathrm{Cl}(2) \cong \mathbb{H}, \quad \mathrm{Cl}(3) \cong \mathbb{H} \oplus \mathbb{H}, \quad \mathrm{Cl}(4) \cong \mathbb{H}(2), \quad \mathrm{Cl}(5) \cong \mathbb{C}(4),$$
$$\mathrm{Cl}(6) \cong \mathbb{R}(8), \quad \mathrm{Cl}(7) \cong \mathbb{R}(8) \oplus \mathbb{R}(8), \quad \mathrm{Cl}(8) \cong \mathbb{R}(16), \quad d_1 = 2,$$
$$d_2 = 4, \quad d_3 = 4, \quad d_4 = 8, \quad d_5 = 8, \quad d_6 = 8, \quad d_7 = 8, \quad d_8 = 16,$$

where $K(n)$ denote the algebra of $n \times n$-matries with entries in $K = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$.

For $n > 8$, *i.e.*, $n = m + 8k(m, k \geq 1)$, $d_{m+8k} = 2^{4k} d_m$.

From Theorem 5.7 and Proposition 5.8, we have the following theorem (cf. [7]).

Theorem 5.9. *Let* $(M, g_M, \mathcal{F})$ *be a Riemannian manifold with a transverse spin foliation* $\mathcal{F}$ *of codimension* $q = 3, 4, 7, 8$ *and a bundle-like metric* g_M *with* $\kappa \in \Omega^1_{\mathrm{B}}(\mathcal{F})$. *Assume that the mean curvature* κ *of* $\mathcal{F}$ *satisfies* $\delta\kappa = 0$ *and* $\sigma^\nabla \geq 0$. *If there exists an eigenspinor field* Ψ_1 *for* λ_1 *with* $\lambda_1^2 = (q/4(q-1))(\mu_1 + \inf|\kappa|^2)$, *then*:

(1) $\mathcal{F}$ *is minimal, transversally Einsteinian.*
(2) $\mathcal{F}$ *is transversally isometric to the action of discrete subgroup of* $\mathrm{O}(q)$ *acting on the* q*-sphere, where* $q = 3, 4, 7, 8$.

Proof. (1) is trivial from Theorem 5.2. Next, we prove (2). Since $\mathcal{F}$ is minimal, from (5.6) we have that $\rho^\nabla(X) = (1/q)\mu_1 X$. Let Ψ and Φ be the foliated spinors with $D_{\mathrm{b}}\Psi = \lambda_1\Psi$ and $D_{\mathrm{b}}\Phi = -\lambda_1\Phi$. From (5.9), we have the following equations. For any $X \in \Gamma Q$

$$\nabla_X \Psi = -\frac{\lambda_1}{q} X \cdot \Psi, \qquad \nabla_X \Phi = \frac{\lambda_1}{q} X \cdot \Phi. \tag{5.16}$$

If we put $f = (\Psi, \Phi)$, then by direct calculation, we have

$$\Delta_{\mathrm{B}} f = \frac{\mu_1}{q-1} f. \tag{5.17}$$

It is sufficient to prove that f does not vanish identically.

(1) In case $q = 4, 8$, it is well known [11] that the real spinor bundle $S(\mathcal{F})$ splits as the two irreducible real representations:

$$S(\mathcal{F}) = S^+(\mathcal{F}) \oplus S^-(\mathcal{F}). \tag{5.18}$$

Then $\Psi = \Psi^+ + \Psi^-$ and $\Phi = \Psi^+ - \Psi^-$, where $\Psi^\pm \in S^\pm(\mathcal{F})$. Hence we have that for any $X \in \Gamma Q$,

$$X(f) = \frac{4\lambda_1}{q}(X \cdot \Psi^+, \Psi^-), \tag{5.19}$$

where $(\,,\,) = \mathrm{Re}\langle\,,\,\rangle$. Let us define the map $F : Q \to S^-(\mathcal{F})$ by $X \to X \cdot \Psi^+$. Then F is the $\mathbb{R}$-linear and injective. Since $d_4 = 8$ and $d_8 = 16$ from Proposition 5.8, $\dim_{\mathbb{R}} Q = \dim_{\mathbb{R}} S^-(\mathcal{F})$. Hence F is isomorphism and there exists $X \neq 0$ such that

$$(X \cdot \Psi^+, \psi^-) \neq 0, \tag{5.20}$$

which implies that $f \not\equiv 0$.

(2) In case $q = 3, 7$, if we define $F : Q \to S(\mathcal{F})$ by $X \to X \cdot \Psi$, then F is $\mathbb{R}$-linear and injective. Since $d_3 = 4$ and $d_7 = 8$ in Proposition 5.8, $\dim_{\mathbb{R}} Q = \dim_{\mathbb{R}} F(Q) = \dim_{\mathbb{R}} S(\mathcal{F}) - 1$. Since $(\Psi, X\cdot\Psi) = 0$, $F(X) \notin E_{\lambda_1}(D_{\mathrm{b}})$, where $E_{\lambda_1}(D_{\mathrm{b}})$ is the eigenspace corresponding to the eigenvalue λ_1. Hence $\dim E_{\lambda_1}(D_{\mathrm{b}}) = 1$ and $F(Q) = E_{\lambda_1}(D_{\mathrm{b}})^\perp$.

So $F : Q \to E_{\lambda_1}(D_b)^{\perp}$ is an isomorphism. Since $\Phi \in F(Q)$, there exists $X \neq 0$ such that

$$X(f) = \frac{-2\lambda_1}{q}(X \cdot \Psi, \Phi) \neq 0, \tag{5.21}$$

which implies that $f \not\equiv 0$. □

Theorem 5.10. *Let* $(M, g_M, \mathcal{F})$ *be a Riemannian manifold with a transverse spin foliation* $\mathcal{F}$ *of codimension* $q = 5$ (*resp.*, $q = 6$) *and a bundle-like metric* g_M. *Assume that the mean curvature* κ *of* $\mathcal{F}$ *satisfies* $\delta\kappa = 0$ *and* $\sigma^{\nabla} \geq 0$. *If the dimension of the eigenspinor space of* λ_1 *with* $\lambda_1^2 = (q/4(q-1))(\mu_1 + \inf|\kappa|^2)$ is 3 (*resp.*, 2), *then*:

(1) $\mathcal{F}$ *is minimal, transversally Einsteinian.*
(2) $\mathcal{F}$ *is transversally isometric to the action of discrete subgroup of* O(5) (*resp.*, O(6)) *acting on the* 5- (*resp.*, 6-)*sphere*.

Proof. The proof is similar to the one of (2) in Theorem 5.9. □

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